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Second quantisation of the nonlinear Schrödinger equation

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Abstract. The classical nonlinear Schrödinger equation may be solved using the inverse scattering transform, but there are difficulties in carrying this over to the case of quantum fields. These difficulties are overcome by explicitly constructing a Fock space representation of the states, together with quantum fields properly defined over this space.

1. Introduction

In the last few years there have been a number of papers concerned with the exact second quantisation of the nonlinear Schrödinger equation (NLSE) using ideas which have emerged from the 'inverse scattering transform' (IST) method (Creamer *et al* 1980, Kaup 1975, Thacker 1978, Thacker and Wilkinson 1979, Sklyanin 1979, Sklyanin and Faddeev 1979, Sklyanin *et al* 1980). This method of solving a large class of nonlinear partial differential equations has been widely investigated and is still the subject of intense research activity. (Many references may be found in Barut (1978).) Two of the classical problems which fall under the ambit of IST are the nonlinear Schrödinger equation and the sine-Gordon equation. These are both of great interest as quantum field equations, and it is no coincidence that the eigenvalues and eigenstates of both problems have been constructed by the Bethe algorithm (Bergknoff and Thacker 1979, McGuire 1964, Yang 1967, 1968), which seems to have some connection with IST (Bergknoff and Thacker 1979, Thacker and Wilkinson 1979).

The earliest paper on the exact quantisation of the NLSE (Kaup 1975) took the viewpoint that IST provides a canonical transformation of the original Hamiltonian system, for which the equation of motion is nonlinear, to a new set of canonical variables in which the system is a collection of non-interacting harmonic oscillators. For the continuous spectrum this provides the correct energy levels, although it is shown below that it does not lead to the correct Hamiltonian operator in terms of the basic 'independent particle' state operators. For the soliton spectrum, canonical quantisation using Kaup's method leads to the wrong energy levels, as Kaup observed by comparing his spectrum with the explicit results of McGuire (1964). Apart from Kaup's work, the methods of IST have been applied to the NLSE in a direct way, and although this shows the obvious promise of the method for quantum field problems, there are some severe difficulties, which were the subject of an earlier paper (Davies 1981). In the present paper these problems are overcome by constructing a Fock space representation using the methods of Glimm and Jaffe (1968, 1970a, b). This is done indirectly, via the Marčenko equation, since it is easier to represent a set of 'independent particle' operators (which satisfy a linear equation but non-canonical quantisation relations) rather than the original quantum fields. The latter are defined by an iteration of the Marčenko equation, and it is shown that they obey the correct equations of motion and commutation relations. However, it is also shown that there are some surprises, in that the 'independent particle' operators behave more like fermion operators than boson operators. The present paper is restricted to the NLSE: investigations for the SG equation are under way, and the results will be published later.

2. Key classical results

The nonlinear Schrödinger equation (in one space dimension) is

$$i\Phi_t = -\Phi_{xx} + 2c^2|\Phi|^2\Phi. \tag{2.1}$$

Solutions of this equation may be found by a number of methods which are all related to IST. Since there are numerous papers on this subject, the only formulae which are included here are those which are directly relevant to the question of second quantisation. In this respect the Marčenko equation is of central importance: for the solution of the nonlinear Schrödinger equation it takes the form

$$K(x, y) = R\left(\frac{x+y}{2}\right) + \frac{c^2}{4}\iint_0^\infty \mathrm{d}s\,\mathrm{d}t\, R^*\left(\frac{2x+s+t}{2}\right)K(x, t)R\left(\frac{s+y}{2}\right). \tag{2.2}$$

Here the function $R(x)$ is a solution of the linear differential equation

$$iR_t = -R_{xx} \tag{2.3}$$

which is the linearised version of the NLSE. In the original IST method, $R(x)$ is constructed from the scattering data. Once the solution of the Marčenko equation has been found using a given $R(x)$, it follows that a solution of the original nonlinear equation is given by

$$\Phi(x) = K(x, x). \tag{2.4}$$

Ablowitz *et al* (1980) have observed that there is no need to appeal to the IST method and the associated Zakharov–Shabat equations in order to construct the function $R(x)$: in fact, equation (2.3) together with some very mild restrictions on the asymptotic form of $R(x)$ is sufficient to guarantee that the Marčenko equation has a solution and that (2.4) is a solution of the NLSE.

It is convenient for the purpose of introducing a Fock space representation of the quantum fields to deal with Fourier transforms. On writing

$$\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx}\,\phi(k) \tag{2.5}$$

and

$$R(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx}\,\rho(k) \tag{2.6}$$

it is possible to rewrite equations (2.1) and (2.3) as

$$\mathrm{i}\phi_t(k)=k^2\phi(k)+\frac{c^2}{2\pi^2}\iint_{-\infty}^{\infty}\mathrm{d}k_1\,\mathrm{d}k_2\,\phi^*(k_1+k_2-k)\phi(k_1)\phi(k_2) \tag{2.7}$$

and

$$\mathrm{i}\rho_t(k)=k^2\rho(k) \tag{2.8}$$

respectively. The Marčenko equation may also be written in Fourier-transform form; this is not done here as there is no need to second quantise the field $K(x, y)$ as a preliminary to quantising $\Phi(x)$. The more direct approach is to solve (2.2) by iteration as a power series in the coupling constant. This was first done by Rosales (1978) using a direct iterative approach; in either case it leads to the expansions

$$\begin{aligned}\Phi(x)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\zeta_1\exp(\mathrm{i}\zeta_1x)\rho(\zeta_1)+\dots\\&+\frac{c^{2n}}{(2\pi)^{2n+1}}\int_{-\infty}^{\infty}\mathrm{d}\xi_1\dots\mathrm{d}\xi_n\,\mathrm{d}\zeta_1\dots\mathrm{d}\zeta_{n+1}\\&\times\exp[-\mathrm{i}(\xi_1+\dots+\xi_n-\zeta_1-\dots-\zeta_{n+1})x]\\&\times\frac{\rho^*(\xi_1)\dots\rho^*(\xi_n)\rho(\zeta_{n+1})\dots\rho(\zeta_1)}{(\zeta_1-\xi_1+\mathrm{i}\varepsilon)(\xi_1-\zeta_2-\mathrm{i}\varepsilon)\dots(\zeta_n-\xi_n+\mathrm{i}\varepsilon)(\xi_n-\zeta_{n+1}-\mathrm{i}\varepsilon)}+\dots\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\phi(k)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\zeta_1\,\delta(k-\zeta_1)\rho(\zeta_1)+\dots\\&+\frac{c^{2n}}{(2\pi)^{2n+1}}\int_{-\infty}^{\infty}\mathrm{d}\xi_1\dots\mathrm{d}\xi_n\,\mathrm{d}\zeta_1\dots\mathrm{d}\zeta_{n+1}\\&\times\delta(k+\xi_1+\dots+\xi_n-\zeta_1-\dots-\zeta_{n+1})\\&\times\frac{\rho^*(\xi_1)\dots\rho^*(\xi_n)\rho(\zeta_{n+1})\dots\rho(\zeta_1)}{(\zeta_1-\xi_1+\mathrm{i}\varepsilon)(\xi_1-\zeta_2-\mathrm{i}\varepsilon)\dots(\zeta_n-\xi_n+\mathrm{i}\varepsilon)(\xi_n-\zeta_{n+1}-\mathrm{i}\varepsilon)}+\dots.\end{aligned} \tag{2.10}$$

These relations are of central importance in this paper.

In making the transition to quantum fields, questions regarding operator ordering are crucial. The appropriate definitions will have to be made in such a way that the quantum field satisfies equation (2.7), and it is necessary to know how this comes about in the classical case. Rosales' method (adapted to the present context) is useful here. On substituting the expansions for $\phi(k)$ into equation (2.7), the terms may be grouped in ascending powers of c^2. The terms in c^{2n} from the linear part of the equation are simply

$$\begin{aligned}&\frac{c^{2n}}{(2\pi)^{2n+1}}\int\mathrm{d}\xi_1\dots\mathrm{d}\xi_n\,\mathrm{d}\zeta_1\dots\mathrm{d}\zeta_{n+1}\,\delta(k+\xi_1+\dots+\xi_n-\zeta_1-\dots-\zeta_{n+1})\\&\quad\times\frac{(\xi_1^2+\dots+\xi_n^2-\zeta_1^2-\dots-\zeta_{n+1}^2+k^2)}{(\zeta_1-\xi_1+\mathrm{i}\varepsilon)(\xi_1-\zeta_2-\mathrm{i}\varepsilon)\dots(\zeta_n-\xi_n+\mathrm{i}\varepsilon)}\\&\quad\times\frac{\rho^*(\xi_1)\dots\rho^*(\xi_n)\rho(\zeta_{n+1})\dots\rho(\zeta_1)}{(\xi_n-\xi_{n+1}-\mathrm{i}\varepsilon)}\end{aligned} \tag{2.11}$$

The nonlinear part of the equation gives many terms, each of the form

$$\frac{2c^{2n}}{(2\pi)^{2n+1}}\int \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_{l_1}\,\mathrm{d}\zeta_1\ldots\mathrm{d}\zeta_{l_1+1}$$

$$\times\mathrm{d}\xi'_1\ldots\mathrm{d}\xi'_{l_2+1}\,\mathrm{d}\zeta'_1\ldots\mathrm{d}\zeta'_{l_2}\,\mathrm{d}\xi''_1\ldots\mathrm{d}\xi''_{l_3}\,\mathrm{d}\zeta''_1\ldots\mathrm{d}\zeta''_{l_3+1}$$

$$\times\frac{\delta(k_1+\xi_1+\ldots+\xi_{l_1}-\zeta_1-\ldots-\zeta_{l_1+1})}{(\zeta_1-\xi_1+\mathrm{i}\varepsilon)(\xi_1-\zeta_2-\mathrm{i}\varepsilon)\ldots(\zeta_{l_1}-\xi_{l_1}+\mathrm{i}\varepsilon)}$$

$$\times\frac{\rho^*(\xi_1)\ldots\rho^*(\xi_{l_1})\rho(\zeta_{l_1+1})\ldots\rho(\zeta_1)}{(\xi_{l_1}-\zeta_{l_1+1}-\mathrm{i}\varepsilon)}$$

$$\times\frac{\delta(k_1+k_2+\xi'_1+\ldots+\xi'_{l_2+1}-\zeta'_1-\ldots-\zeta'_{l_2})}{(\xi'_1-\zeta'_1-\mathrm{i}\varepsilon)(\zeta'_1-\xi'_2+\mathrm{i}\varepsilon)\ldots(\xi_{l_2}-\zeta_{l_2}-\mathrm{i}\varepsilon)}$$

$$\times\frac{\rho^*(\xi'_1)\ldots\rho^*(\xi'_{l_2+1})\rho(\zeta'_{l_2})\ldots\rho(\zeta'_1)}{(\zeta'_{l_2}-\xi'_{l_2+1}+\mathrm{i}\varepsilon)}$$

$$\times\frac{\delta(k_2+\xi''_1+\ldots+\xi''_{l_3}-\zeta''_1\ldots-\zeta''_{l_3+1})}{(\zeta''_1-\xi''_1+\mathrm{i}\varepsilon)(\xi''_1-\zeta''_2-\mathrm{i}\varepsilon)\ldots(\zeta''_{l_3}-\xi''_{l_3}+\mathrm{i}\varepsilon)}$$

$$\times\frac{\rho^*(\xi''_1)\ldots\rho^*(\xi''_{l_3})\rho(\zeta''_{l_3+1})\ldots\rho(\zeta''_1)}{(\xi''_{l_3}-\zeta''_{l_3+1}-\mathrm{i}\varepsilon)} \tag{2.12}$$

where

$$l_1+l_2+l_3=n-1. \tag{2.13}$$

On performing the integrals over k_1 and k_2, and carrying out the following relabelling:

$$\begin{aligned}&(\xi_1\ldots\xi_{l_1},\xi'_1\ldots\xi'_{l_2+1},\xi''_1\ldots\xi''_{l_3})\rightarrow(\xi_1\ldots\xi_n),\\&(\zeta_1\ldots\zeta_{l_1+1},\zeta'_1\ldots\zeta'_{l_2},\zeta''_1\ldots\zeta''_{l_3+1}\}\rightarrow(\zeta_1\ldots\zeta_{n+1}),\end{aligned} \tag{2.14}$$

this term becomes almost identical with (2.11). The only difference is that instead of the factor

$$[\xi_1^2+\ldots+\xi_n^2-\zeta_1^2-\ldots-\zeta_{n+1}^2+(\xi_1+\ldots+\xi_n-\zeta_1-\ldots-\zeta_{n+1})^2] \tag{2.15}$$

there now appears the factor

$$2(\xi_{l_1+1}-\zeta_{l_1+1})(\xi_{l_1+l_2+1}-\zeta_{l_1+l_2+2}). \tag{2.16}$$

However, the sum of these factors over all partitions satisfying (2.13) is simply the negative of (2.15), so that the expansion (2.9) satisfies the nonlinear Schrödinger equation to every order of c^2. This result is no surprise once one knows of IST; what is important is that the relabelling (2.14) is imposed by the algebra and it contains essential information regarding the ordering of operators in Fock space.

3. Fock space representation

Field quantisation is most readily effected by introducing a Fock representation, using the beautifully clear notation of Glimm and Jaffe (1970a). An element in this Fock space H is an infinite sequence

$$(\psi_0, \psi_1(k_1), \ldots, \psi_n(k_1, \ldots, k_n), \ldots) \tag{3.1}$$

where $\psi_n(k_1, \ldots, k_n)$ is a symmetric function of its variables. The basic operators which act in the space are $\rho(k)$ and $\rho^+(k)$. In terms of the scattering data $a(k)$ and $b(k)$ of IST, they are given by

$$\rho(k) = b^*(k)/a^*(k), \qquad \rho^+(k) = b(k)/a(k), \tag{3.2}$$

when the convention of Creamer *et al* (1980) is used to write $\rho^+(k)$ as the creation operator. These authors give the commutation relations as

$$\rho(k)\rho(k') = S(k', k)\rho(k')\rho(k), \tag{3.3}$$

$$\rho(k)\rho^+(k') = S(k, k')\rho^+(k')\rho(k) + 2\pi\delta(k-k'), \tag{3.4}$$

$$S(k, k') = (k - k' - \mathrm{i}c^2)/(k - k' + \mathrm{i}c^2). \tag{3.5}$$

It might be thought that these relations could be obtained by first calculating the classical Poisson brackets and then making the appropriate identification with quantum commutators. There is no difficulty with the first step (Zakharov and Manakov 1975): the result is

$$\begin{aligned} \{\rho(k), \rho(k')\} &= -[2c^2/(k-k')]\rho(k)\rho(k'), \\ \{\rho(k), \rho^*(k')\} &= [2c^2/(k-k')]\rho(k)\rho^*(k') - 2\pi\mathrm{i}\delta(k-k'). \end{aligned} \tag{3.6}$$

Provided that $k \neq k'$, this gives the correct commutation relations; however, there is no way of obtaining the correct form of (3.4) from (3.6) in the neighbourhood of $k = k'$. Indeed, in this neighbourhood, the operators have the commutation relations

$$\begin{aligned} \rho(k)\rho(k') + \rho(k')\rho(k) &\simeq 0, \\ \rho(k)\rho^+(k') + \rho^+(k')\rho(k) &\simeq 2\pi\delta(k-k'), \end{aligned} \tag{3.7}$$

which are appropriate for fermions. An examination of the work of McGuire (1964) on the exact solution of the n-particle Schrödinger equation with delta function interaction shows that this surprising result is correct. In fact, he showed that the n-particle states are uniquely labelled by n quantum numbers which may be chosen independently except that no two may coincide. Moreover, he showed that, while the $c \to 0$ limit does not exist, for $c \to \infty$ the solution varies continuously toward the solution of a non-interacting fermion problem.

Returning to the actual representation of the commutation relations in Fock space, an examination of known results for the effect of the operators on the n-particle Bethe eigenstates suggests that the appropriate form is

$$(\rho(k)\psi)_{n-1}(k_1, \ldots, k_{n-1}) = (2\pi n)^{1/2} \prod_{j=1}^{n-1} f(k, k_j)\psi_n(k, k_1, \ldots, k_{n-1}) \tag{3.8}$$

and

$$(\rho^+(k)\psi)_{n+1}(k_1, \ldots, k_{n+1}) = \left(\frac{2\pi}{(n+1)}\right)^{1/2} \sum_{i=1}^{n+1} \delta(k-k_i) \prod_{j \neq i} f^*(k, k_j)\psi_n(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{n+1}). \tag{3.9}$$

It is customary to refer to $\rho(k)$ and $\rho^+(k)$ as operators, and we shall adhere to that usage. However, $\rho^+(k)$ is certainly not an operator in the strict sense of mapping a subspace of H into another subspace. It is, however, a perfectly well defined distribution on a suitable nuclear subspace Ω of H. A convenient choice for this nuclear subspace is the set of finite sequences of functions, each function being a test function of fast decrease in its n variables (Gelfand and Shilov 1964, Gelfand and Vilenkin 1964).

The weights $f(k, k')$ are readily determined. First, note that there is no loss of generality involved in using $f^*(k, k')$ in (3.9): it is necessary in order to ensure that the relation

$$\langle\phi, \rho(k)\psi\rangle = \langle\rho^+(k)\phi, \psi\rangle^* \tag{3.10}$$

holds. It is readily checked that (3.8) and (3.9) will reproduce the commutation relations (3.3) and (3.4) only if

$$|f(k, k')| = 1, \qquad f(k, k')/f(k', k) = S(k, k'). \tag{3.11}$$

The unique solution of these conditions is

$$f(k, k') = \exp[\mathrm{i}\theta(k-k')] \tag{3.12}$$

where

$$\theta(k-k') = \tan^{-1}[(k-k')/c^2], \qquad 0 \leqslant \theta \leqslant \pi. \tag{3.13}$$

Returning to the earlier discussion of the choice of commutation relations, it is interesting to note that equations (3.3), (3.8) and (3.9) suffice to determine (3.11), removing all freedom in choosing (3.4). This is interesting in its own right, since (3.3) can be found by the 'Poisson bracket–commutator' argument, whereas (3.4) cannot.

Quantum fields may now be defined in an unambiguous way. First, the unbounded operators $R(x)$ and $R^+(x)$ are defined as

$$R(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\zeta\, \mathrm{e}^{\mathrm{i}\zeta x}\, \rho(\zeta), \tag{3.14}$$

$$R^+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-\mathrm{i}\xi x}\, \rho^+(\xi). \tag{3.15}$$

It is interesting to observe the effect of $R^+(x)$ on the vacuum vector, which will be denoted by ψ_0 throughout the following. A simple inductive argument shows that

$$\begin{aligned}(R^+(x_n)\ldots R^+(x_1)\psi_0)_n(k_1,\ldots,k_n) \\ &= (2\pi)^{-n/2}(n!)^{-1/2}\sum_{P[n]}\exp[-\mathrm{i}(k_{p_1}x_1+\ldots+k_{p_n}x_n)] \\ &\quad\times\prod_{i>j}\exp[-\mathrm{i}\theta(k_{p_i}-k_{p_j})]\end{aligned} \tag{3.16}$$

where $P[n]$ is a permutation of the integers $1, \ldots, n$. These functions are identical with the Bethe eigenstates for the corresponding n-particle Schrödinger equation with one exception: they are correctly normalised, whereas the usual Bethe eigenstates are not. Thus the results which emerge from the representation (3.8)–(3.9) are not identical with those obtained by Creamer *et al* (1980), although the differences are not great.

For any test function $f(x)$, the bounded operators $R(f)$ and $R^+(f)$ are

$$R(f)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta\, \tilde{f}(\zeta)\rho(\zeta), \tag{3.17}$$

$$R^+(f)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, \tilde{f}(-\xi)\rho^+(\xi), \tag{3.18}$$

where $\tilde{f}$ is the Fourier transform of f. The fields $R(f)$ and $R^+(f)$ have, for a minimum common domain of definition, the nuclear subspace Ω of finite sequences of test functions. Furthermore, the effect of polynomials of the creation operators acting on the vacuum vector is readily calculated as

$$(R^+(f_n)\ldots R^+(f_1)\psi_0)_n(k_1,\ldots,k_n)$$
$$=(2\pi)^{-n/2}(n!)^{-1/2}\sum_{P[n]}\tilde{f}(-k_{p_1})\ldots\tilde{f}(-k_{p_n})\prod_{i>j}\exp[-i\theta(k_{p_i}-k_{p_j})]. \tag{3.19}$$

This is a set of functions dense in H, so the fields are complete and the vacuum vector cyclic (Bogolubov *et al* 1975).

4. The quantum fields $\Phi(x)$ and $\phi(k)$

The fields which were constructed in the previous section are pertinent to the Marčenko equation rather than the original NLSE. However, they may be substituted into the formulae (2.9) and (2.10) in order to define the quantum fields $\Phi(x)$ and $\phi(k)$. It is apparent from the commutation relations (3.3) and (3.4) that this must be done very carefully, since the basic annihilation and creation operators $\rho(k)$ and $\rho^+(k)$ do not commute among themselves. Although the considerations of § 2 were restricted to classical fields, the orderings given in equations (2.9) and (2.10) are those which give rise to the desired results. The main purpose of the remainder of this paper is to prove this assertion: certainly it cannot be taken for granted simply because the Marčenko equation solves the classical NLSE.

The immediate task is to check that the quantum fields $\Phi(x)$ and $\Phi^+(x)$ satisfy the canonical commutation relations

$$[\Phi(x),\Phi(x')]=0, \tag{4.1}$$

$$[\Phi(x),\Phi^+(x')]=\delta(x-x'), \tag{4.2}$$

or

$$[\Phi(f),\Phi(g)]=0, \tag{4.3}$$

$$[\Phi(f),\Phi^+(g)]=\int_{-\infty}^{\infty} dx\, f(x)g(x). \tag{4.4}$$

This is necessary because (3.3) and (3.4) were introduced as a postulate, not deduced from the canonical commutation relations.

It is readily shown that

$$\Phi^+(x)\psi_0=R^+(x)\psi_0; \tag{4.5}$$

furthermore, it is shown in the Appendix that

$$\Phi^+(x)R^+(x') = R^+(x')\Phi(x) \qquad (x < x'). \tag{4.6}$$

From these two relations it follows by an inductive argument that

$$\Phi^+(x_n)\ldots\Phi^+(x_1)\psi_0 = R^+(x_{p_n})\ldots R^+(x_{p_1})\psi_0 \qquad (x_{p_n} > x_{p_{n-1}} > \ldots > x_{p_1}) \tag{4.7}$$

and

$$\Phi(f_n)\ldots\Phi(f_1)\psi_0 = \int dx_1\ldots dx_n\, f_n(x_n)\ldots f_1(x_1) \times \sum_{P[n]} \theta(x_{p_n} > \ldots > x_{p_1}) R^+(x_{p_n})\ldots R^+(x_{p_1})\psi_0 \tag{4.8}$$

which are equivalent to the commutation relations (4.1) and (4.3) respectively, since the vacuum vector is cyclic. Similarly, it is shown in the Appendix that

$$\Phi(f)R^+(x_n)\ldots R^+(x_1)\psi_0 = \sum_j f(x_j)R^+(x_n)\ldots R^+(x_{j+1})R^+(x_{j-1})\ldots R^+(x_1)\psi_0 \times (x_n > x_{n-1} > \ldots > x_1) \tag{4.9}$$

which implies the commutation relations (4.2) and (4.4). One of the remarkable features of the manipulations which lead to these results is the way in which horrendously complicated algebraic expressions simplify. While it is true that the fields $R(x)$ satisfy an uncomplicated linear equation of motion (2.3), this is at the expense of introducing rather complicated commutation relations (3.3) and (3.4).

5. Normal ordered polynomials

It is apparent from the commutation relations (3.3) and (3.4) that the definition of normal ordering of polynomials in $\Phi(x)$ and $\Phi^+(x)$ must be made very carefully when equation (2.9) is used, since the operators $\rho(k)$ and $\rho(k')$ do not commute, even though $\Phi(x)$ and $\Phi(x')$ do. The appropriate choice is to require that the normal ordering should preserve the original order of each of the two kinds of operator $\rho(k)$ and $\rho^+(k)$, except for placing all creation operators to the left of annihilation operators. For example,

$$:\rho^+(\xi_1)\rho(\zeta_2)\rho^+(\xi_2)\rho^+(\xi_3)\rho(\zeta_1): = \rho^+(\xi_1)\rho^+(\xi_2)\rho^+(\xi_3)\rho(\zeta_2)\rho(\zeta_1). \tag{5.1}$$

Using this procedure, normal ordered polynomials of the quantum fields may be defined. An important case is

$$\Phi^+(x)\Phi(x)\Phi(x) = \frac{1}{2\pi}\int dk\, e^{ikx} \frac{1}{(2\pi)^2}\int dk_1\, dk_2\, \phi^+(k_1+k_2-k)\phi(k_1)\phi(k_2) \tag{5.2}$$

and this may be written in terms of $\rho(k)$ and $\rho^+(k)$ by substituting equation (2.10) and then normal ordering every term in the resulting expansion. With this definition, the arguments embodied in equations (2.11)–(2.16) carry through to the second quantised form and the quantum fields satisfy the NLSE.

Conserved quantities are of great interest in the classical field case, in particular the normalisation, momentum and energy functionals:

$$N=\int \mathrm{d}x\ \Phi^{+}(x)\Phi(x), \tag{5.3}$$

$$P=-\mathrm{i}\int \mathrm{d}x\ \Phi^{+}(x)\Phi_x(x), \tag{5.4}$$

$$H=\int \mathrm{d}x[\Phi_x^{+}(x)\Phi_x(x)+2c^2\Phi^{+}(x)\Phi^{+}(x)\Phi(x)\Phi(x)]. \tag{5.5}$$

These functionals may be expressed in terms of the new canonical variables provided by IST as (Kaup 1975)

$$N=\frac{1}{2\pi}\int \mathrm{d}k\ \ln[1+\rho^{+}(k)\rho(k)], \tag{5.6}$$

$$P=\frac{1}{2\pi}\int \mathrm{d}k\ k\ \ln[1+\rho^{+}(k)\rho(k)], \tag{5.7}$$

$$H=\frac{1}{2\pi}\int \mathrm{d}k\ k^2\ \ln[1+\rho^{+}(k)\rho(k)], \tag{5.8}$$

and it may be shown that H is the Hamiltonian for the classical NLSE.

It is immediately evident that equations (5.6)–(5.8) do not carry over to the quantum field in the obvious manner. For example, equations (3.8) and (3.9) show that

$$\begin{aligned}\int \mathrm{d}x\ (R^{+}(x)R(x)\psi)_n(k_1,\ldots,k_n)\\ &=\frac{1}{2\pi}\int \mathrm{d}k(\rho^{+}(k)\rho(k)\psi)_n(k_1,\ldots,k_n)\\ &=\sum_{i=1}^{n}\int \mathrm{d}k\ \delta(k-k_i)\psi_n(k, k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_n)\\ &=n\psi_n(k_1,\ldots,k_n)\end{aligned} \tag{5.9}$$

so that $R^{+}(x)R(x)$ is the number operator, whereas (5.6) suggests that $\ln[1+R^{+}(x)R(x)]$ is the number operator. Similarly,

$$\frac{1}{2\pi}\int \mathrm{d}k\ k^m(\rho^{+}(k)\rho(k)\psi)_n(k_1,\ldots,k_n)=\sum_{i=1}^{n}k_i^m\psi_n(k_1,\ldots,k_n). \tag{5.10}$$

so that the Hamiltonian operator has to be

$$H=\frac{1}{2\pi}\int \mathrm{d}k\ k^2\rho^{+}(k)\rho(k) \tag{5.11}$$

since the energy levels of the Bethe eigenstates are $\Sigma\, k_i^2$. This operator, although not bounded over the entire Hilbert space H, is certainly bounded over all but two of the infinite sequence of subspaces which come from completing Ω in the countable sequence of norms (Gelfand and Vilenkin 1964)

$$\|\psi\|_n = \sum_{m=0}^{\infty} \int dk_1 \ldots dk_m\, (k_1^n + \ldots + k_m^n)|\psi_m(k_1, \ldots, k_m)|^2. \tag{5.12}$$

Although the identification of (5.11) with the normal ordered form of (5.5) has been made here on the basis of comparing the known spectra of the two operators, a direct proof of the equivalence of these two operators over Ω may be written out using equation (2.7) and the results of § 4. Alternatively, one may observe that

$$\begin{aligned}[\rho(k), H] &= \frac{1}{2\pi} \int d\xi\, \xi^2 [\rho(k), \rho^+(\xi)\rho(\xi)] \\ &= k^2 \rho(k) \\ &= i\rho_t(k)\end{aligned} \tag{5.13}$$

from which it follows, by substituting into equation (2.9), that

$$i\frac{\partial\Phi(x)}{\partial t} = [\Phi(x), H]. \tag{5.14}$$

Thus H generates the correct equation of motion for the field operators.

6. Conclusions

It is interesting to recall the results of Glimm and Jaffe (1970a, b) for the relativistic ϕ^4 theory in one space dimension. Their proof of the existence of a Hilbert space in which the renormalised Hamiltonian and field operators are properly defined, with a unique vacuum state, covers more than 60 pages in addition to using other theorems which they published separately. They commence by setting up a Fock space realisation of the commutation relations (4.7) and (4.8) in conjunction with the linearised equation of motion: this space is totally inadequate once the interaction is included, and their constructions involve taking inductive limits. The main result of the present paper is to construct a simple Fock space representation of a Hilbert space suitable for the full nonlinear problem. The enormous difference between this space and a Hilbert space which is suitable for the linearised problem is seen by comparing the commutation relations (3.7)—which are closer to anticommutation relations—with (4.1)–(4.2). This also gives a clue as to why there are difficulties with the direct use of IST for a quantum field. The Zakharov–Shabat equations are (Zakharov and Shabat 1975)

$$v_{1x} - \tfrac{1}{2}i\xi v_1 = icv_2\psi, \qquad v_{2x} + \tfrac{1}{2}i\xi v_2 = -ic\psi^* v_1, \tag{6.1}$$

and from these all of the necessary dynamical variables are defined. However, for quantum fields, the problem arises that ψ and v_i act in Hilbert spaces with totally different structures. Certainly it is not adequate simply to state that the fields are operators and then proceed as though they had a workable common domain of definition.

Investigations are under way to obtain similar results to the above for the sine-Gordon equation, which is also soluble classically using IST. The quantum SG equation, for bosons, is known to be equivalent to the massive Thirring model for fermions:

moreover, this latter model has recently been exactly diagonalised by Bergknoff and Thacker (1979) using the Bethe ansatz. All of these facts point to the probability that the present methods will shed considerable light on both of these problems.

Appendix

To show that

$$\Phi^+(x)R^+(x') = R^+(x')\Phi^+(x) \qquad (x<x') \tag{A1}$$

consider the general term which results from substituting the expansion (2.9) for $\Phi^+(x)$ in the left-hand side, namely

$$\begin{aligned}\frac{c^{2n}}{(2\pi)^{2n+2}}\int & \mathrm{d}\xi_1\ldots\mathrm{d}\xi_{n+1}\,\mathrm{d}\zeta_1\ldots\mathrm{d}\zeta_n\,\mathrm{d}\xi \\ &\times\exp[-\mathrm{i}(\xi_1+\ldots+\xi_{n+1}-\zeta_1-\ldots-\zeta_n)x-\mathrm{i}\xi x'] \\ &\times\frac{\rho^+(\xi_1)\ldots\rho^+(\xi_{n+1})\rho(\zeta_n)\ldots\rho(\zeta_1)\rho^+(\xi)}{(\xi_1-\zeta_1-\mathrm{i}\varepsilon)(\zeta_1-\xi_2+\mathrm{i}\varepsilon)\ldots(\xi_n-\zeta_n-\mathrm{i}\varepsilon)(\zeta_n-\xi_{n+1}+\mathrm{i}\varepsilon)}.\end{aligned} \tag{A2}$$

In order to bring this to the form of the right-hand side it is necessary to commute the operator $\rho^+(\xi)$ through all of the operators $\rho(\zeta_j)$. This gives a term involving $\delta(\zeta_j-\xi)$ for each j, and one more term in which $\rho^+(\xi)$ survives. Explicitly, this last term is

$$\begin{aligned}\frac{c^{2n}}{(2\pi)^{2n+2}}\int & \mathrm{d}\xi\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_{n+1}\,\mathrm{d}\zeta_1\ldots\mathrm{d}\zeta_n \\ &\times\exp[-\mathrm{i}\xi x'-\mathrm{i}(\xi_1+\ldots+\xi_{n+1}-\zeta_1-\ldots-\zeta_n)x] \\ &\times S(\xi,\xi_1)\ldots S(\xi,\xi_{n+1})S(\zeta_1,\xi)\ldots S(\zeta_n,\xi) \\ &\times\frac{\rho^+(\xi)\rho^+(\xi_1)\ldots\rho^+(\xi_{n+1})\rho(\zeta_n)\ldots\rho(\zeta_1)}{(\xi_1-\zeta_1-\mathrm{i}\varepsilon)(\zeta_1-\xi_2+\mathrm{i}\varepsilon)\ldots(\xi_n-\zeta_n-\mathrm{i}\varepsilon)(\zeta_n-\xi_{n+1}+\mathrm{i}\varepsilon)}\end{aligned} \tag{A3}$$

while the terms which involve delta functions are

$$\begin{aligned}\frac{c^{2n}}{(2\pi)^{2n+1}}\int & \mathrm{d}\xi_1\ldots\mathrm{d}\xi_{n+1}\,\mathrm{d}\zeta_1\ldots\mathrm{d}\zeta_n\,\mathrm{d}\xi \\ &\times\exp[-\mathrm{i}(\xi_1+\ldots+\xi_{n+1}-\zeta_1-\ldots-\zeta_n)x-\mathrm{i}\xi x'] \\ &\times S(\zeta_1,\xi)\ldots S(\zeta_{j-1},\xi)\delta(\zeta_j-\xi) \\ &\times\frac{\rho^+(\xi_1)\ldots\rho^+(\xi_{n+1})\rho(\zeta_n)\ldots\rho(\zeta_{j+1})\rho(\zeta_{j-1})\ldots\rho(\zeta_1)}{(\xi_1-\zeta_1-\mathrm{i}\varepsilon)(\zeta_1-\xi_2+\mathrm{i}\varepsilon)\ldots(\xi_n-\zeta_n-\mathrm{i}\varepsilon)(\zeta_n-\xi_{n+1}+\mathrm{i}\varepsilon)}.\end{aligned} \tag{A4}$$

Equation (A3) differs from the corresponding term of the right-hand side of (A1) only through the factor $S(\xi,\xi_1)\ldots S(\xi,\xi_{n+1})S(\zeta_1,\xi)\ldots S(\zeta_n,\xi)$. In order to compensate for this difference, it is necessary to combine (A3) (with n reduced to $n-1$ so that there are $n+1$ creation operators and n annihilation operators) with the terms (A4) for $j=1,\ldots,n$. Equation (A4) must therefore be brought to the same order, with the creation operator associated with x' to the left of all the other operators. Also, it is necessary to integrate out the variables ξ and ζ_j which no longer label operators, using the formula

$$\int d\xi \frac{\exp[i\xi(x-x')]f(\xi)}{(\xi_1-\xi-i\varepsilon)(\xi-\xi_2+i\varepsilon)} = \frac{2\pi i}{\xi_1-\xi_2}\{\exp[i\xi_1(x-x')]f(\xi_1)-\exp[i\xi_2(x-x')]f(\xi_2)\}. \quad \text{(A5)}$$

For each j, there are two terms, namely

$$\begin{aligned}
&\frac{ic^{2n}}{(2\pi)^{2n}}\int d\xi_1\ldots d\xi_{n+1}\, d\zeta_1\ldots d\zeta_{j-1}\, d\zeta_{j+1}\ldots d\zeta_n \\
&\qquad\times\exp[-i(\xi_1+\ldots+\xi_j+\xi_{j+2}+\ldots+\xi_{n+1} \\
&\qquad-\zeta_1-\ldots-\zeta_{j-1}-\zeta_{j+1}-\ldots-\zeta_n)x-i\xi_{j+1}x'] \\
&\qquad\times S(\xi_{j+1},\xi_1)\ldots S(\xi_{j+1},\xi_j)S(\zeta_1,\xi_{j+1})\ldots S(\zeta_{j-1},\xi_{j+1}) \\
&\qquad\times\frac{\rho^+(\xi_{j+1})\rho^+(\xi_1)\ldots\rho^+(\xi_j)\rho^+(\xi_{j+2})\ldots\rho^+(\xi_{n+1})}{(\xi_{j+1}-\xi_j)(\xi_1-\zeta_1-i\varepsilon)\ldots(\zeta_{j-1}-\xi_j+i\varepsilon)} \\
&\qquad\times\frac{\rho(\zeta_n)\ldots\rho(\zeta_{j+1})\rho(\zeta_{j-1})\ldots\rho(\zeta_1)}{(\xi_{j+1}-\zeta_{j+1}-i\varepsilon)\ldots(\zeta_n-\xi_{n+1}+i\varepsilon)} \\
&\quad=\frac{ic^{2n}}{(2\pi)^{2n}}\int d\xi\, d\xi_1\ldots d\xi_n\, d\zeta_1\ldots d\zeta_{n-1} \\
&\qquad\exp[-i\xi x'-i(\xi_1+\ldots+\xi_n-\zeta_1-\ldots-\zeta_{n-1})x] \\
&\qquad\times S(\xi,\xi_1)\ldots S(\xi,\xi_j)S(\zeta_1,\xi)\ldots S(\zeta_{j-1},\xi) \\
&\times\frac{\rho^+(\xi)\rho^+(\xi_1)\ldots\rho^+(\xi_n)}{(\xi-\xi_j)(\xi_1-\zeta_1-i\varepsilon)\ldots(\zeta_{j-1}-\xi_j+i\varepsilon)}\,\frac{\rho(\zeta_{n-1})\ldots\rho(\zeta_1)}{(\xi-\zeta_j-i\varepsilon)(\zeta_j-\xi_{j+1}+i\varepsilon)\ldots(\zeta_{n-1}-\xi_n+i\varepsilon)}
\end{aligned} \quad \text{(A6)}$$

and

$$\begin{aligned}
&\frac{ic^{2n}}{(2\pi)^{2n}}\int d\xi_1\ldots d\xi_{n+1}\, d\zeta_1\ldots d\zeta_{j-1}\, d\zeta_{j+1}\ldots d\zeta_n \\
&\qquad\times\exp[-i(\xi_1+\ldots+\xi_{j-1}+\xi_{j+1}+\ldots+\xi_n \\
&\qquad-\zeta_1-\ldots-\zeta_{j-1}-\zeta_{j+1}-\ldots-\zeta_n)x-i\xi_j x'] \\
&\qquad\times S(\xi_j,\xi_1)\ldots S(\xi_j,\xi_{j-1})S(\zeta_1,\xi_j)\ldots S(\zeta_{j-1},\xi_j) \\
&\qquad\times\frac{\rho^+(\xi_1)\ldots\rho^+(\xi_{n+1})\rho(\zeta_n)\ldots\rho(\zeta_{j+1})}{(\xi_j-\xi_{j+1})(\xi_1-\zeta_1-i\varepsilon)\ldots(\zeta_{j-1}-\xi_j+i\varepsilon)} \\
&\qquad\times\frac{\rho(\zeta_{j-1})\ldots\rho(\zeta_1)}{(\xi_j-\zeta_{j+1}-i\varepsilon)\ldots(\zeta_n-\xi_{n+1}+i\varepsilon)} \\
&\quad=\frac{ic^{2n}}{(2\pi)^{2n}}\int d\xi\, d\xi_1\ldots d\xi_n\, d\zeta_1\ldots d\zeta_{n-1} \\
&\qquad\times\exp[-i\xi x'-i(\xi_1+\ldots+\xi_n-\zeta_1-\ldots-\zeta_{n-1})x] \\
&\qquad\times S(\xi,\xi_1)\ldots S(\xi,\xi_{j-1})S(\zeta_1,\xi)\ldots S(\zeta_{j-1},\xi) \\
&\qquad\times\frac{\rho^+(\xi)\rho^+(\xi_1)\ldots\rho^+(\xi_n)\rho(\zeta_{n-1})\ldots\rho(\zeta_1)}{(\xi-\xi_j)(\xi_1-\zeta_1-i\varepsilon)\ldots(\zeta_j-\xi+i\varepsilon)(\xi_j-\zeta_j-i\varepsilon)\ldots(\zeta_{n-1}-\xi_n+i\varepsilon)}.
\end{aligned} \quad \text{(A7)}$$

The second form of each of these expressions is the result of further relabelling of variables aimed at making the ensuing algebra easier. Taking the term with $j=n$ from (A6) and (A7) and combining with (A3)—the latter with n replaced by $n-1$—leads to the following weight factor:

$$\frac{c^{2n-2}}{(2\pi)^{2n}}S(\xi,\xi_1)\ldots S(\xi,\xi_{n-1})S(\zeta_1,\xi)\ldots S(\zeta_{n-1},\xi)$$

$$\times\left(S(\xi,\xi_n)+\frac{\mathrm{i}c^2}{\xi-\xi_n}S(\xi,\xi_n)+\frac{\mathrm{i}c^2(\zeta_{n-1}-\xi_n)}{(\xi-\xi_n)(\zeta_{n-1}-\xi)}\right)$$

$$=\frac{c^{2n-2}}{(2\pi)^{2n}}S(\xi,\xi_1)\ldots S(\xi,\xi_{n-1})S(\zeta_1,\xi)\ldots S(\zeta_{n-2},\xi)\frac{\xi-\zeta_{n-1}+\mathrm{i}c^2}{\xi-\zeta_{n-1}}. \tag{A8}$$

Combining now with the $j=n-1$ term gives

$$\frac{c^{2n-2}}{(2\pi)^{2n}}S(\xi,\xi_1)\ldots S(\xi,\xi_{n-2})S(\zeta_1,\xi)\ldots S(\zeta_{n-3},\xi)\frac{\xi-\zeta_{n-2}+\mathrm{i}c^2}{\xi-\zeta_{n-2}} \tag{A9}$$

and, continuing in this way, the $j=1$ term gives the desired result, namely

$$\frac{c^{2n-2}}{(2\pi)^{2n}}\left(S(\xi,\xi_1)\frac{\xi-\zeta_1+\mathrm{i}c^2}{\xi-\zeta_1}+\frac{\mathrm{i}c^2(\xi_1-\zeta_1)}{(\xi-\xi_1)(\xi-\zeta_1)}S(\xi,\xi_1)+\frac{\mathrm{i}c^2}{\xi-\xi_1}\right)=\frac{c^{2n-2}}{(2\pi)^{2n}}. \tag{A10}$$

Thus the identity (A1) is proved to each order in the expansion in powers of c. One of the remarkable features of this algebra is that it depends on ordering the operators in $\rho(x)$ and $\rho^+(x)$ in precisely the manner indicated in equation (2.9). For example, it may be readily checked that (A1) is invalid if the annihilation operators are written in the reverse order.

To prove (4.15) it is convenient first to prove that, if $f(x)$ is of bounded support, then

$$\Phi(f)R^+(x_n)\ldots R^+(x_1)\psi_0$$

$$=[f(x_n)+R^+(x_n)]\Phi(f)R^+(x_{n-1})\ldots R^+(x_1)\psi_0 \qquad (x_n>x_{n-1}>\ldots>x_1) \tag{A11}$$

from which the result follows immediately by induction. Substituting the expansion (2.9) for $\Phi(x)$, the lowest-order term on the left-hand side of (A11) is

$$\frac{1}{(2\pi)^2}\int \mathrm{d}x\,\mathrm{d}\zeta\,\mathrm{d}\xi_1\,f(x)\exp(\mathrm{i}\zeta x-\mathrm{i}\xi_1x_n)\rho(\zeta)\rho^+(\xi_1)R^+(x_{n-1})\ldots R^+(x_1)\psi_0$$

$$=\frac{1}{(2\pi)^2}\int \mathrm{d}x\,\mathrm{d}\xi_1\,\mathrm{d}\zeta\,f(x)\exp(-\mathrm{i}\xi_1x_n+\mathrm{i}\zeta x)$$

$$\times[2\pi\delta(\zeta-\xi_1)+S(\zeta,\xi_1)\rho^+(\xi_1)\rho(\zeta)]R^+(x_{n-1})\ldots R^+(x_1)\psi_0$$

$$=f(x_n)R^+(x_{n-1})\ldots R^+(x_1)\psi_0$$

$$+\frac{1}{(2\pi)^2}\int \mathrm{d}x\,\mathrm{d}\xi_1\,\mathrm{d}\zeta\,f(x)\exp(-\mathrm{i}\xi_1x_n+\mathrm{i}\zeta x)$$

$$\times S(\zeta,\xi_1)\rho^+(\xi_1)\rho(\zeta)R^+(x_{n-1})\ldots R^+(x_1)\psi_0. \tag{A12}$$

Thus the first term emerges in a trivial fashion. The remaining terms depend on algebraic manipulations which are entirely parallel to those used in proving (A1). For these terms, it is necessary to split up the integral in (A12) into the regions $x > x_n$ and $x < x_n$. In the first region, the inequalities $x_n > x_{n-1} > \ldots > x_1$, together with the fact that $\tilde{f}$ is an entire function of exponentially bounded growth in its imaginary part, guarantee that every term is zero. In the region $x < x_n$, the integral (A5) may be used as before, and this part of $\Phi(f)$ commutes with $R^+(x_n)$ to complete the demonstration of (A11).

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